



1. An algebra teacher, testing her students on geometric sequences, asked her students to find the sum of all three terms of a geometric sequence in which the middle number was missing. Unfortunately, one student confused geometric sequences with arithmetic sequences, and then completed the problem with no other errors, obtaining an answer of 351. If all numbers involved were distinct positive integers, compute, with proof, the correct answer to the teacher's question.

2. The numbers  $x_1, x_2, x_3, \dots, x_n$  are written on a chalkboard.  $-2(kboa)-6(r)3(d.0+941o8d1-oa)-4haooo8$

# SOLUTIONS–KSU MATHEMATICS COMPETITION – PART II 2009-10

1. Represent the terms of the geometric series with  $ar^2$ , Then the student's sequence becomes  $\frac{1}{2}(a - ar^2)$ ,  $ar^2$ . Thus,  $a - \frac{1}{2}(a - ar^2) = ar^2 + 351$ .

Simplifying,  $3a - 3ar^2 = 702$ . Solving this last equation for  $a$ ,  $\frac{234}{1 - r^2}$ .

Noting that  $234 = (2)(3)(13)$ , this equation will only yield integral values of  $a$  for  $r = 1$  and  $r = 5$ . If  $r = 1$ , the numbers in the teacher's sequence are not distinct. Therefore,  $r = 5$  and the three numbers are 9, 45, and 225 with a sum of 279.

2. Proof by mathematical induction on the number of numbers,  $n$ .

If  $n = 1$ , we certainly have  $(1 - x_1) = 1 - x_1$ .

Suppose the statement is true for all  $k < n$  numbers. Suppose the process has repeated until only two numbers,  $a$  and  $b$ , are left, and suppose  $a$  was obtained by combining  $x_1, x_2, \dots, x_m$  and  $b$  was obtained by combining  $x_{m+1}, x_{m+2}, \dots, x_k$ . Then by the induction hypothesis,

$$a = (1 - x_1)(1 - x_2) \dots (1 - x_m) \cdot 1 \text{ and } b = (1 - x_{m+1})(1 - x_{m+2}) \dots (1 - x_k) \cdot 1.$$

Therefore, the final number is given by  $a + b + ab =$

$$\begin{aligned} a + b + ab &= (1 - x_1)(1 - x_2) \dots (1 - x_k) + \underbrace{(1 - x_1)(1 - x_2) \dots (1 - x_m)}_{a} + \underbrace{(1 - x_{m+1})(1 - x_{m+2}) \dots (1 - x_k)}_{b} + 1 = \\ &= a + b + (1 - x_1)(1 - x_2) \dots (1 - x_k) + (a + b) = \\ &= (1 - x_1)(1 - x_2)(1 - x_3) \dots (1 - x_k) + 1 \end{aligned}$$

3.  $\log_{\sin x}(\tan x) = \log_{\sin x}(\sin x) = \frac{1}{\log_{\sin x}(\tan x)}$  (by the change of base formula),

where  $\sin x > 0$  and  $\tan x > 0$ . Therefore,  $(\log_{\sin x}(\tan x))^2 = 1$ ,

which implies  $\log_{\sin x}(\tan x) = \pm 1$ .

$$\log_{\sin x}(\tan x) = 1 \longrightarrow \tan x = \sin x \longrightarrow \cos x = 1.$$

However, this would make  $\sin x = 0$ .

$$\log_{\sin x}(\tan x) = -1 \longrightarrow \tan x = \frac{1}{\sin x} \longrightarrow \cos x = \sin x = 1 - \cos^2 x$$

$$\text{Therefore, } \cos^2 x + \cos x - 1 = 0 \longrightarrow \cos x = \frac{1 - \sqrt{5}}{2}.$$

If  $\cos x$  is negative, then one of  $\tan x$  or  $\sin x$  is negative also. Therefore, the only possible value of  $\cos x$  is  $\frac{1 - \sqrt{5}}{2}$ .

4. Let  $m \angle P = x$  and  $m \angle PAB = m \angle PAD = y$ .  
 Since  $\angle PBC$  is an exterior angle of triangle  $PAB$ ,  
 $m \angle PBC = x + y$  and  $m \angle DBP = x + y$  also.  
 Therefore,  $m \angle DBC = 2x + 2y$ . Since  $\angle DBC$  is  
 an exterior angle of triangle  $ABD$ ,  
 $m \angle DBC = m \angle D + m \angle DAB$  or  
 $2x + 2y = m \angle D + 2y$ .  
 Therefore,  $m \angle D = 2x$  and the desired ratio is 2:1.

5. Let the area of the circle  $T$  be equal to 1.  
 Denote by  $A_j$  the  $j$ -th patch and by  $P_j = |A_j|$  the area of  $j$ -th patch.  
 Analogously we define  $P_{ijk}$ ,  $P_{ijkl}$ , and  $P$