1. Find, with proof, all integers n such that $2^6 + 2^9 + 2^n$ is the square of an integer.

2. Find, with proof, all real numbers a such that $|x-1| - |x-2| + |x-4| = a$ has exactly 3 solutions.

3. As Lisa hurried to copy down the last problem of her math homework assignment at the end of class, she got as far as

$$0 = 9x^8 - 28x^6 -$$

SOLUTIONS

1. If $m^2 = 2^6 + 2^9 + 2^n = 576 + 2^n = 24^2 + 2^n$, then $2^n = m^2 - 24^2 = (m - 24)(m + 24)$. Therefore, m + 24 and m – 24 must each be powers of 2. Let
m + 24 = 2^k and m – 24 = 2^p where, p < k and p + k = n. Then
$2^k - 2^p = 48$ which implies 2^p divides 48, so that $p \le 4$. Trying p = 0, 1, 2, 3, 4 gives
p = 4, k = 6 and n = 10 as the only possible value for n.

2. We begin by graphing the function y = f(x) = $|x - 1| - |x - 2| + |x - 4|$. If $x \le 1$,
then $x - 1 \le 0$, $x - 2 \le 0$ and $x - 4 \le 0$, so we have

$$y = -(x - 1) + (x - 2) - (x - 4) = -x + 3.$$

If $1 \le x \le 2$, then then $x - 1 \ge 0$, $x - 2 \le 0$ and $x - 4 \le 0$, so

$$y = (x - 1) + (x - 2) - (x - 4) = x + 1.$$

In the interval $2 \le x \le 4$ we have $x - 1 \ge 0$, $x - 2 \ge 0$ but $x - 4 \le 0$, so

$$y = (x - 1) - (x - 2) - (x - 4) = -x + 5.$$

Finally, for $x \ge 4$ we have

$$y = (x - 1) - (x - 2) + (x - 4) = x - 3.$$

By piecing together the relevant parts of these four
linear functions, we get the graph of the function f(x)
shown at the right.

Thus, for the original equation to have exactly three solutions, we have to choose
a so that the horizontal line y = a touches the graph of f(x) at exactly three points.
From the graph, this happens only for a = 2 and a = 3.

3. P(x) = $9x^8 - 28x^6 - \ldots$ Since the coefficient of x^7 is zero, the sum of the roots of
P(x) = 0 is zero. Let the seven identical roots be a and the desired eighth root be b.
Then 7a + b = 0 or b = -7a. Standardizing P(x), the sum of the products of the roots
taken two at a time is $-\frac{28}{9}$. Therefore, C ^{2} 7 $\frac{28}{9}$

4\. a. Let the desired pair-square set be {2012, a, b}. Since $2012 + 13 = 2025 = 45^2$, try $a = 13$ as a second member of the set. Then

$$2012 + b = x^2 \quad \text{and} \quad 13 + b = y^2 \quad \text{for some integers } x \text{ and } y.$$

Subtracting these two equations gives $1999 = x^2 - y^2 = (x + y)(x - y)$.
Since 1999 is a prime number, $x + y = 1999$ and $x - y = 1$. From these two equations we obtain $x = 1000$. Then $b = 1000^2 - 2012 = 1000000 - 2012 = 997988$.
Thus, one desired pair-square set of size 3 is {13, 2012, 997988}.

(Note: {292, 2012, 45077} and {488, 2012, 143912} also work. There are others.)

b. Every integer has one of the four forms $4k$; $4k + 1$; $4k + 2$ and $4k + 3$ for integers k.

First we prove that the square of an integer must have one of the forms $4n$ or $4n + 1$.

Proof:

(i) $(4k)^2 = 4(4k^2) = 4n$

(ii) $(4k+1)^2 = 16k^2 + 8k + 1 = 4(4k \quad + 2k) + 1 = 4n + 1$

(iii) $(4k+2)^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1) = 4n$

(iv) $(4k+3)^2 = 16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1 = 4n + 1$

Therefore, the square of an integer must have one of the forms $4n$ or $4n + 1$.

Let S be a pair-square set of size 3. Suppose that the set S contains the two odd numbers a and b. Since $a + b$ is an even square, it must have form $4n$, and therefore a and b cannot both have form $4k + 1$, nor can they both have form $4k + 3$. It follows that we can write $a = 4k + 1$ and $b = 4k + 3$.

We derive a contradiction by showing that there is no possibility for the third member z of S. Indeed, if z has form 4k or $4k + 3$, then $z + b$ is not a square, and if z has form 4k + 4b is Td (k)Tj /b Td (.-8.93 -1.10 Td (a)Tj /TT0 1 Tf [(+)4( )]TJ2(4)4TJ -0.)

5. It is easy to show that ΔADP is isosceles (note the marked congruent angles).

Thus DP = 2012 and all we need is the length of PC.
Since ΔADP and ΔCBQ are congruent isosceles triangles,
$\angle PAB \cong \angle DAP \cong \angle CQB$, making $\overline{AP}$ parallel to $\overline{QC}$.
Similarly, the other two angle bisectors are parallel, making EFGH a parallelogram.

Since $\angle DAB$ and $\angle ADC$ are supplementary, $m\angle DAB + m\angle ADC = 180$.

Since m